We have $M+N=4$. According to AM-GM, we get

$$\begin{aligned} M{+}S &= \frac{a+b}{b+c}+\frac{b+c}{c+d}+\frac{c+d}{d+a}+\frac{d+a}{a+b}\geq 4; \\ N{+}S &= \frac{a+c}{b+c}+\frac{b+d}{c+d}+\frac{a+c}{d+a}+\frac{b+d}{a+b} \\ &= \frac{a+c}{b+c}+\frac{a+c}{a+d}+\frac{b+d}{c+d}+\frac{b+d}{a+b} \\ &\geq \frac{4(a+c)}{a+b+c+d}+\frac{4(b+d)}{a+b+c+d}=4. \end{aligned}$$

Therefore, $M+N+2S\geq 8$, and $S\geq 2$. The equality holds if $a=b=c=d$ or $a=c, b=d=0$ or a=c=0, b=d.

Also solved by Arkady Alt, San Jose, CA; D.M. Bătinetu-Giurgiu, Bucharest, Neculai Stanciu Buzău and Titu Zvonaru Comănesti, all from Romania (two solutions); Bruno Salgueiro Fanego, Viveiro, Spain; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Adrian Naco, Polytechnic University, Tirana, Albania; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain, and the proposer.

- **5211:** *Proposed by Ovidiu Furdui, Cluj-Napoca, Romania*

Let $n\geq 1$ be a natural number and let

$$f_n(x)=x^{x^{\cdot^{\cdot^{\cdot^{x}}}}},$$

where the number of x's in the definition of f_n is n. For example

$$f_1(x)=x, \quad f_2(x)=x^x, \quad f_3(x)=x^{x^x}, \ldots .$$

Calculate the limit

$$\lim_{x\to 1}\frac{f_n(x)-f_{n-1}(x)}{(1-x)^n}.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

We show the limit equals $(-1)^n$. Define $f_0(x)=1$. For $n\geq 2$ and $x>0$, we have $f_n(x)=e^{f_{n-1}(x)\ln x}$. Hence by the mean value theorem, we have

$$f_n(x)-f_{n-1}(x)=\ln x\,(f_{n-1}(x)-f_{n-2}(x))\,e^{\xi},$$

where ξ lies between $f_{n-1}(x)\ln x$ and $f_{n-2}(x)\ln x$.

Since $\lim\limits_{x\to 1}f_{n-1}(x)\ln x=\lim\limits_{x\to 1}f_{n-2}(x)\ln x=0$ and $\lim\limits_{x\to 1}\dfrac{\ln x}{1-x}=-1$, so

$$\lim_{x\to 1}\frac{f_n(x)-f_{n-1}(x)}{(1-x)^n}=-\lim_{x\to 1}\frac{f_{n-1}(x)-f_{n-2}(x)}{(1-x)^{n-1}}.$$

Clearly $\lim_{x\to 1}\dfrac{f_1(x)-f_0(x)}{1-x}=-1$. Hence by induction we have

$$\lim_{x\to 1}\frac{f_n(x)-f_{n-1}(x)}{(1-x)^n}=(-1)^n,$$

as claimed.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

We will use induction to prove that $a_n=\lim_{x\to 1}\dfrac{f_n(x)-f_{n-1}(x)}{(1-x)^n}=(-1)^n$.
We have by applying L'Hôpital's rule twice,

$$a_2=\lim_{x\to 1}\frac{f_2(x)-f_1(x)}{(1-x)^2}=\lim_{x\to 1}\frac{x^x-x}{(1-x)^2}=\lim_{x\to 1}\frac{x^x(1+\log x)-1}{-2(1-x)}=\lim_{x\to 1}\frac{x^x\left[(1+\log x)^2+\frac{1}{x}\right]}{2}=1.$$

So the assertion holds for $n=2$.
We have $\dfrac{d}{dx}f_n(x)=\dfrac{d}{dx}e^{f_{n-1}(x)\log x}=f_n(x)\left(f'_{n-1}(x)\log x+\dfrac{f_{n-1}(x)}{x}\right)$. In particular,

$$f'_n(1)=f_n(1)\left(f'_{n-1}(1)\log(1)+\frac{f_{n-1}(1)}{1}\right)=1.$$

So, by L'Hôpital's rule,

$$\begin{aligned}
a_n &= \lim_{x\to 1}\frac{f_n(x)-f_{n-1}(x)}{(1-x)^n}=\lim_{x\to 1}\frac{f'_n(x)-f'_{n-1}(x)}{-n(1-x)^{n-1}}\\
&= \lim_{x\to 1}\frac{f_n(x)\left(f'_{n-1}(x)\log x+\frac{f_{n-1}(x)}{x}\right)-f_{n-1}(x)\left(f'_{n-2}(x)\log x+\frac{f_{n-2}(x)}{x}\right)}{-n(1-x)^{n-1}}\\
&= \lim_{x\to 1}\frac{\left(f_n(x)-(f_{n-1}(x)\right)\left(f'_{n-1}(x)\log x+\frac{f_{n-1}(x)}{x}\right)}{-n(1-x)^{n-1}}\\
&+ \lim_{x\to 1}\frac{\left(f_{n-1}(x)\left(f'_{n-1}(x)\log x+\frac{f_{n-1}(x)}{x}-f'_{n-2}(x)\log x-\frac{f_{n-2}(x)}{x}\right)\right.}{-n(1-x)^{n-1}}.
\end{aligned}$$

So

$$\begin{aligned}
a_n &= \lim_{x\to 1}\frac{f_n(x)-f_{n-1}(x)}{(1-x)^n}\left(1+\frac{\left(f'_{n-1}(x)\log x+\frac{f_{n-1}(x)}{x}\right)(1-x)}{n}\right)\\
&= \lim_{x\to 1}\frac{f_{n-1}(x)\left(f'_{n-1}(x)\log x+\frac{f_{n-1}(x)}{x}-f'_{n-2}(x)\log x-\frac{f_{n-2}(x)}{x}\right)}{-n(1-x)^{n-1}}
\end{aligned}$$

$$= \lim_{x\to 1} \frac{f'_{n-1}(x) - f'_{n-2}(x)}{-n(1-x)^{n-2}} \cdot \frac{\log x}{1-x} + \lim_{x\to 1} \frac{f_{n-1}(x) - f_{n-2}(x)}{-n(1-x)^{n-1}x}$$

$$= \frac{n-1}{n} \lim_{x\to 1} \frac{f_{n-1}(x) - f_{n-2}(x)}{(1-x)^{n-1}} \cdot (-1) + \frac{1}{(-n)} \lim_{x\to 1} \frac{f_{n-1}(x) - f_{n-2}(x)}{(1-x)^{n-1}}$$

$$= -a_{n-1} = -(-1)^{n-1} = (-1)^n.$$

Solution 3 by Adrian Naco, Polytechnic University, Tirana, Albania

At first we observe that the function is of the form

$$f_n(x) = x^{f_{n-1}(x)} = e^{f_{n-1}(x)\ln x}$$

and that

$$\begin{aligned}\frac{f_n(x) - f_{n-1}(x)}{(1-x)^n} &= \frac{e^{f_{n-1}(x)\ln x} - e^{f_{n-2}(x)\ln x}}{(1-x)^n} = e^{f_{n-2}(x)\ln x}\left(\frac{e^{[f_{n-1}(x)-f_{n-2}(x)]\ln x} - 1}{(1-x)^n}\right) \\ &= e^{f_{n-2}(x)\ln x} \cdot \frac{[f_{n-1}(x) - f_{n-2}(x)]\ln x}{(1-x)^n} \cdot \frac{e^{[f_{n-1}(x)-f_{n-2}(x)]\ln x} - 1}{[f_{n-1}(x) - f_{n-2}(x)]\ln x}. \qquad (2)\end{aligned}$$

The function $f_1(x) = x$ is continuous everywhere for $x > 0$ and

$$\lim_{x\to 1} f_1(x) = \lim_{x\to 1} x = 1.$$

One easily comes to the conclusion that the function $f_n(x) = e^{f_{n-1}(x)\ln x}$ is continuous everywhere for $x > 0$ as a composition of a product of two continuous functions $u(x) = f_{n-1}(x)\ln x$ and the exponential function $f_n(x) = e^{u(x)}$ and as a logical result implies that

$$\lim_{x\to 1} f_n(x) = e^{\lim\limits_{x\to 1}[f_{n-1}(x)\ln x]} = e^{\left[\lim\limits_{x\to 1} f_{n-1}(x)\right]\cdot\left[\lim\limits_{x\to 1}\ln x\right]} = e^{1\cdot 0} = 1. \quad (3)$$

Using the known limit rule

$$\lim_{\alpha\to 0} \frac{e^{\alpha} - 1}{\alpha} = 1 \Rightarrow \lim_{x\to 1} \frac{e^{[f_{n-1}(x)-f_{n-2}(x)]\ln x} - 1}{[f_{n-1}(x) - f_{n-2}(x)]\ln x} = 1 \qquad (4)$$

since

$$\begin{aligned}\lim_{x\to 1} \alpha(x) &= \lim_{x\to 1} [f_{n-1}(x) - f_{n-2}(x)]\ln x \\ &= \left[\lim_{x\to 1} f_{n-1}(x) - \lim_{x\to 1} f_{n-2}(x)\right]\left(\lim_{x\to 1}\ln x\right) \\ &= (1-1)\cdot 0 = 0\end{aligned}$$

So from formula (2) and (4) we derive the inductive result for one step.

$$\begin{aligned}
& \lim_{x\to 1} \frac{f_n(x) - f_{n-1}(x)}{(1-x)^n} \\
= \; & \lim_{x\to 1} e^{f_{n-2}(x)\ln x} \cdot \frac{[f_{n-1}(x) - f_{n-2}(x)]\ln x}{(1-x)^n} \cdot \frac{e^{[f_{n-1}(x)-f_{n-2}(x)]\ln x} - 1}{[f_{n-1}(x) - f_{n-2}(x)]\ln x} \\
= \; & \left(\lim_{x\to 1} e^{f_{n-2}(x)\ln x}\right) \cdot \left(\lim_{x\to 1} \frac{[f_{n-1}(x) - f_{n-2}(x)]\ln x}{(1-x)^n}\right) \cdot \left(\lim_{x\to 1} \frac{e^{[f_{n-1}(x)-f_{n-2}(x)]\ln x} - 1}{[f_{n-1}(x) - f_{n-2}(x)]\ln x}\right) \\
= \; & \left(e^{\lim_{x\to 1} f_{n-2}(x)\ln x}\right) \cdot \left(\lim_{x\to 1} \frac{[f_{n-1}(x) - f_{n-2}(x)]\ln x}{(1-x)^n}\right) \cdot \left(\lim_{x\to 1} \frac{e^{[f_{n-1}(x)-f_{n-2}(x)]\ln x} - 1}{[f_{n-1}(x) - f_{n-2}(x)]\ln x}\right) \\
= \; & 1 \cdot \left(\lim_{x\to 1} \frac{[f_{n-1}(x) - f_{n-2}(x)]\ln x}{(1-x)^n}\right) \cdot 1 = \lim_{x\to 1} \frac{[f_{n-1}(x) - f_{n-2}(x)]\ln x}{(1-x)^n} \qquad (5)
\end{aligned}$$

Inductively we derive the general formula

$$\begin{aligned}
\lim_{x\to 1} \frac{f_n(x) - f_{n-1}(x)}{(1-x)^n} &= \lim_{x\to 1} \frac{f_n(x) - f_{n-1}(x)}{(1-x)^n} \ln^0 x \\
&= \lim_{x\to 1} \frac{[f_{n-1}(x) - f_{n-2}(x)]\ln^1 x}{(1-x)^n} \\
&= \lim_{x\to 1} \frac{[f_{n-2}(x) - f_{n-3}(x)]\ln^2 x}{(1-x)^n} \\
& \cdots\cdots\cdots \\
&= \lim_{x\to 1} \frac{[f_2(x) - f_1(x)]\ln^{n-2} x}{(1-x)^n} = \lim_{x\to 1} \frac{[x^x - x]\ln^{n-2} x}{(1-x)^n} \\
&= \lim_{x\to 1} \frac{\left[e^{x\ln x} - e^{\ln x}\right]\ln^{n-2} x}{(1-x)^n} = \lim_{x\to 1} \frac{e^{\ln x}\left[e^{(x-1)\ln x} - 1\right]\ln^{n-2} x}{(1-x)^n} \\
&= \lim_{x\to 1} e^{\ln x} \frac{\left[e^{(x-1)\ln x} - 1\right]}{(x-1)\ln x}(x-1)\frac{\ln^{n-1} x}{(1-x)^n} \\
&= (-1)\left(\lim_{x\to 1} e^{\ln x}\right)\left(\lim_{x\to 1} \frac{\left[e^{(x-1)\ln x} - 1\right]}{(x-1)\ln x}\right)\left[\lim_{x\to 1} \frac{\ln x}{(1-x)}\right]^{n-1} \\
&= (-1)\cdot e^0 \cdot 1 \cdot \left[\lim_{x\to 1} \frac{\ln x}{(1-x)}\right]^{n-1} = -\left[\lim_{x\to 1} \frac{\ln x}{(1-x)}\right]^{n-1}.
\end{aligned}$$

Applying L'Hôpital's rule we have that

$$\begin{aligned}
\lim_{x\to 1}\frac{f_n(x)-f_{n-1}(x)}{(1-x)^n} &= -\left[\lim_{x\to 1}\frac{\ln x}{(1-x)}\right]^{n-1} = -\left[\lim_{x\to 1}\frac{(\ln x)'}{(1-x)'}\right]^{n-1} \\
&= -\left[\lim_{x\to 1}\frac{\frac{1}{x}}{(-1)}\right]^{n-1} = (-1)(-1)^{n-1} = (-1)^n.
\end{aligned}$$

Editor's comment: There was a mistake in the statement of the problem when it first appeared on the web. That version asked for the $\lim_{x\to 1}\frac{f_n(x)-f_{n-1}(x)}{(1-x)^{n+1}}$. *This mistake was corrected almost immediately but not before a few of the readers started working with the incorrect statement of the problem; although those readers noted the error and corrected it in their solutions, once again, mea culpa. Most all who submitted solutions to this problem approached it with induction.*

Also solved by Arkady Alt, San Jose, CA; Anastasios Kontronis, Athens, Greece; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy, and the proposer.